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SECOND VARIATIONS FOR THE STRESS WAVE PROBLEM USING THE EULER-LAGRANGE AND ADJOINT FORMULATIONS

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The adjoint system can be arranged in a manner so it is a reflected mirror of the original system in time. Generalized boundary conditions employ many types of "springs" relating the various spatial partial derivatives. They are defined to satisfy the boundaries of the original and adjoint system relationship for the bilinear expression. Algorithms for use in the finite element method are simplified since the adjoint system gives exactly the same solutions
(CONT'D ON REVERSE)

20. ABSTRACT (CONT'D)

as those of the original system. The second necessary condition for an extremum is satisfied by showing that the second variation of the functional is positive semi-definite.

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INTRODUCTION

Dynamical stress behaviors and shock transients in mechanics are important subjects to be studied. The use of the finite element method based on developed algorithms from the variational principle can give direct numerical solutions for partial derivatives of the functions to these problems.

Many researchers have found it difficult to apply the finite element method to hyperbolic type partial differential equations (PDE). Galerkin method and the like have been used instead. Previously, the author has attempted to solve these hyperbolic type PDE's by employing variational principles. The present report shows that under certain conditions, the stationary values are strong extremals, not saddle points. This is equivalent to requiring that the second variations of the functional be positive semi-definite with discontinuities in the partials.

THE FUNCTIONAL BILINEAR FORM

A functional in bilinear form (refs 1,2) is assumed for the derivation of the Euler-Lagrange equations of the original system and its symmetrical adjoint system so that the first variations of the functional can be used for numerical computation by the finite element method (ref 3). Let us consider the functional

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[1] C. N. Shen, "Method of Solution for Variational Principle Using Bicubic Hermite Polynomial," Transactions of the 27th Conference of Army Mathematicians, ARO Report 82-1, p. 247.

[2] C. N. Shen, "Variational Principle for Gun Dynamics With Adjoint Variable Formulation," Proceedings of the Third US Army Symposium on Gun Dynamics, Volume II, 1982, pp. IV108-120.

[3] A. R. Mitchell and R. Wait, The Finite Element Method in Partial Differential Equations, John Wiley, 1977, p. 143.

$$J = \int_{t_o}^{t_b} \int_{x_o}^{t_b} F[t,x,y,y_x,y_t,y_{xx},y_{tt},\bar{y},\bar{y}_x,\bar{y}_t,\bar{y}_{xx},\bar{y}_{tt}]dxdt \tag{1}$$

where $y(x,t)$ is the original variable, $\bar{y}(x,t)$ is the adjoint variable, and the subscripts x and t indicate the partial derivatives.

The variations of the variables are in terms of a small parameter μ as

$$y(x,t,\mu) = y(x,t) + \delta y(x,t) \quad , \quad \delta y(x,t) = \mu\eta(x,t) \tag{2a}$$

$$y_x(x,t,\mu) = y_x(x,t) + \delta y_x(x,t) \quad , \quad \delta y_x(x,t) = \mu\eta_x(x,t) \tag{2b}$$

and

$$y_t(x,t,\mu) = y_t(x,t) + \delta y_t(x,t) \quad , \quad \delta y_t(x,t) = \mu\eta_t(x,t) \tag{2c}$$

Similar expressions can be obtained for higher derivatives of y.

The functions $y(x,t,\mu)$ and $y(x,t)$ are close or neighboring in the sense of closeness (ref 4) of order one if the following holds for some $\varepsilon > 0$

$$|\mu\eta(x,t)| < \varepsilon \quad , \quad |\mu\eta_x(x,t)| < \varepsilon \quad , \quad \text{and } |\mu\eta_t(x,t)| < \varepsilon \tag{3a}$$

It is noted from above that the partial derivatives of the variations are small for a weak extremum.

On the other hand, for a strong extremum we require only

$$|\mu\eta(x,t)| < \varepsilon \tag{3b}$$

The adjoint variable $\bar{y}(x,t,\mu)$ and its partials can be obtained by adding a bar on top of the original variable y and its partials as given by Eq. (2).

THE FIRST VARIATION AND THE EULER-LAGRANGE EQUATIONS

The functional $J(\mu)$ in Eq. (1) can be expanded in a Taylor series (ref 5) of μ as

$$J(\mu) = J(\mu=0) + \delta J + \delta^2 J \tag{4a}$$

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[4]L. E. Elsgolc, Calculus of Variations, Addison-Wesley, 1962, p. 15.
[5]H. Rund, The Hamilton Jacob Theory of the Calculus of Variations, Robert E. Krieger, 1973, pp. 5-8.

where

$$\delta J = \mu\left(\frac{\delta J}{\partial u}\right)_{\mu=0} \tag{4b}$$

and

$$\delta^2 J = \frac{\mu^2}{2}\left(\frac{\partial^2 J}{\partial u^2}\right)_{\mu=0} \tag{4c}$$

Equation (1) involves two independent variables x and t, two dependent variables y and $\bar{y}$, and each variable has two second order derivatives with respect to x and t. The first variation can be deduced from Gelfand and Formin (ref 6) and separated into two parts as

$$\delta J = \delta J(\delta y) + \delta J(\delta \bar{y}) \tag{5a}$$

where

$$\delta J(\delta y) = \int_{t_o}^{t_b}\int_{x_o}^{x_b}\Big(F_y - \frac{\partial}{\partial t}F_{y_x} - \frac{\partial}{\partial x}F_{y_x} + \frac{\partial^2}{\partial t^2}F_{y_{tt}} + 2\frac{\partial^2}{\partial t\,\partial x}F_{y_{tt}} + \frac{\partial^2}{\partial x^2}F_{y_{xx}}\Big)\delta y(x,t)\,dxdt \tag{5b}$$

and

$$\delta J(\delta \bar{y}) = \int_{t_o}^{t_b}\int_{x_o}^{x_b}\Big(F_{\bar{y}} - \frac{\partial}{\partial t}F_{\bar{y}_t} - \frac{\partial}{\partial x}F_{\bar{y}_x} + \frac{\partial^2}{\partial t^2}F_{\bar{y}_{tt}} + 2\frac{\partial^2}{\partial t\,\partial x}F_{\bar{y}_{tx}} + \frac{\partial^2}{\partial x^2}F_{y_{xx}}\Big)\delta \bar{y}(x,t)\,dxdt \tag{5c}$$

For the wave equation let us consider the following F in Eq. (1)

$$F = \frac{1}{2}\bar{y}(y_{tt}-a^2y_{xx}) + \bar{y}Q + \frac{1}{2}y(\bar{y}_{tt}-a^2\bar{y}_{xx}) + y\bar{Q} \tag{6}$$

By evaluating the terms in Eq. (5b) for the above F, we have

$$\delta J(\delta y) = \int_{t_o}^{t_b}\int_{x_o}^{x_b}\bar{\phi}\;\delta y\;dxdt \tag{7a}$$

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[6] I. M. Gelfand and S. V. Formin, <u>Calculus of Variations</u>, Prentice-Hall, 1963, pp. 22, 34, 42, and 161.

where

$$\bar{\phi} = \bar{y}_{tt} - a^2\bar{y}_{xx} + \bar{Q} \tag{7b}$$

Similarly by Eq. (5c) one obtains

$$\delta J(\delta\bar{y}) = \int_{t_o}^{t_b} \int_{x_o}^{x_b} \phi \; \delta\bar{y} \; dxdt \tag{7c}$$

where

$$\phi = y_{tt} - a^2 y_{xx} + Q \tag{7d}$$

The first variations of δy and $\delta\bar{y}$ are arbitrary within certain limitations to be discussed later. Then the fundamental lemma (ref 7) of calculus of variation gives

$$\bar{\phi} = \bar{y}_{tt} - a^2\bar{y}_{xx} + \bar{Q} = 0 \tag{8a}$$

and

$$\phi = y_{tt} - a^2 y_{xx} + Q = 0$$

which are the adjoint and the original systems of the wave equation. Note that the above two equations are similar and interchangeable by adding or dropping the bar on top of y.

THE SECOND VARIATIONS

In order that the functional J in Eq. (1) be an extremum (ref 8), the second variation $\delta^2 J$ must be either positive semi-definite for a minimum, (or negative semi-definite for a maximum), i.e.,

$$\delta^2 J \geq 0 \quad (\text{or } \delta^2 J \leq 0) \tag{9}$$

The above is the second necessary condition for a minimum (or a maximum).

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[7] R. Weinstock, Calculus of Variations, McGraw-Hill, 1952, p. 16.
[8] Hans Sagan, Introduction to the Calculus of Variations, McGraw-Hill, 1969, p. 38.

From Eqs. (4b), (7a), and (7c), one obtains

$$\frac{\delta J}{\partial \mu} = \int_{t_o}^{t_b} \int_{x_o}^{x_b} (\phi \; \delta\bar{\eta} + \bar{\phi} \; \delta\eta) dxdt \tag{10}$$

where

$$\phi = Q + \mu\eta_{tt} - a^2\mu\eta_{xx} \tag{11a}$$

and

$$\bar{\phi} = \bar{Q} + \mu\bar{\eta}_{tt} - a^2\mu\bar{\eta}_{xx} \tag{11b}$$

$$\frac{\partial^2 J}{\partial \mu^2} = \int_{t_o}^{t_b} \int_{x_o}^{x_b} \{[\frac{\partial x}{\partial y}\frac{\partial y}{\partial \mu} + \frac{\partial x}{\partial y_t}\frac{\partial y_t}{\partial \mu} + \frac{\partial x}{\partial y_x}\frac{\partial y_x}{\partial \mu}$$

$$+ \frac{\partial x}{\partial y_{tt}}\frac{\partial y_{tt}}{\partial \mu} + 2\frac{\partial x}{\partial y_{xx}}\frac{\partial y_{tx}}{\partial \mu} + \frac{\partial x}{\partial y_{xx}}\frac{\partial y_{xx}}{\partial \mu}] d\bar{\eta}$$

$$+ \text{(similar term with bars)} d\eta\} dxdt \tag{12}$$

$$= \int_{t_o}^{t_b} \int_{x_o}^{x_b} \{(\eta_{tt} - a^2\eta_{xx})\bar{\eta} + (\bar{\eta}_{tt} - a^2\bar{\eta}_{xx})\eta\} dxdt \tag{13}$$

Integrated by parts we have

$$\delta^2 J = \frac{\mu^2}{2}\left(\frac{\partial^2 J}{\partial \mu^2}\right) = \int_{t_o}^{t_b} \int_{x_o}^{x_b} \mu^2[-\eta_t(x,t)\bar{\eta}_t(x,t) + a^2\eta_x(x,t)\bar{\eta}_x(x,t)] dxdt$$
$$+ \text{B.C.} + \text{I.C.} \tag{14}$$

It can be proved that for physical problems such as a fixed or a free end on either side, the boundary conditions are zero. The initial conditions are zero as illustrated in the next section.

THE ADJOINT SYSTEM

The adjoint system may be taken as the image reflection in the time domain of the original system, as shown in Figure 1.

$$\bar{y}(x,t) = y(x,t_b+t_o-t) \tag{15a}$$

$$\bar{y}_x(x,t) = y_x(x,t_b+t_o-t) \tag{15b}$$

$$\bar{y}_t(x,t) = -y_t(x,t_b+t_o-t) \tag{15c}$$

In addition, Eq. (15) yields the known initial conditions as

$$\bar{y}(x,t_b) = y(x,t_o) \tag{16a}$$

$$\bar{y}_t(x,t_b) = -y_t(x,t_o) \tag{16b}$$

Since the adjoint system is a reflected mirror in time, the far end initial conditions for the adjoint system are known.

We may now derive the variations of Eqs. (15) and (16), which give

$$\delta\bar{y}(x,t) = \delta y(x,t_b+t_o-t) \tag{17a}$$

$$\delta\bar{y}_x(x,t) = \delta y_x(x,t_b+t_o-t) \tag{17b}$$

$$\delta\bar{y}_t(x,t) = -\delta y_t(x,t_b+t_o-t) \tag{17c}$$

$$\delta\bar{y}(x,t_b) = \delta y(x,t_o) = 0 \quad \text{for all } x \tag{18a}$$

and

$$\delta\bar{y}_x(x,t_b) = -\delta y_t(x,t_o) = 0 \quad \text{for all } x \tag{18b}$$

By substituting Eq. (15) into Eq. (14) and using Eq. (2), we have

$$\delta^2 J = \int_{t_o}^{t_b} \int_{x_o}^{x_b} P(x,t)\,dx\,dt \tag{19a}$$

where

$$\begin{aligned} P(x,t) &= \delta y_t(x,t)\,\delta y_t(x,t_b+t_o-t) + a^2 \delta y_x(x,t)\,\delta y_x(x,t_b+t_o-t) \\ &= \mu^2[\eta_t(x,t)\,\eta_t(x,t_b+t_o-t) + a^2 \eta_x(x,t)\,\eta_x(x,t_b+t_o-t)] \end{aligned} \tag{19b}$$

SENSITIVITY RELATIONSHIP

In order to show that the second variation of the functional J is positive semi-definite, one needs to obtain the variations of the function and its partials together with those of the adjoint function and its partials as indicated in Eq. (19). We can get these variations through the study of the sensitivity coefficients (ref 9) and their relationship to the parameters

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[9]Rajko Tomovic, Sensitivity Analysis of Dynamic Systems, McGraw-Hill, 1963.

given in Eq. (2). Let the forcing function in Eq. (7d) be

$$Q(x,t) = qf(x,t) \tag{20}$$

It is assumed that the forcing function parameter q is subject to a small constant perturbation δq as

$$q = q_o + \delta q \tag{21}$$

Then the variation of the function y is

$$\delta y(x,t) = \frac{\partial y(x,t)}{\partial q} \delta q = \nu(x,t)\,\delta q \tag{22a}$$

where

$$\nu(x,t) = \frac{\partial y}{\partial q} \tag{22b}$$

The quantity ν is the sensitivity coefficient for the variation $\delta y(x,t)$ due to a small constant perturbation δq.

The original PDE in Eq. (7d) can be written as

$$\begin{aligned} \phi &= Ly + Q \\ &= y_{tt} - a^2 y_{xx} + qf(x,t) = 0 \end{aligned} \tag{23}$$

Due to the perturbation of q, the change of ϕ obeys the following relationship:

$$\frac{\partial \phi}{\partial y_{tt}} \frac{\partial y_{tt}}{\partial q} + \frac{\partial \phi}{\partial y_{xx}} \frac{\partial y_{xx}}{\partial q} + f(x,t) = 0 \tag{24}$$

It is also noted from Eq. (23) that

$$\frac{\partial \phi}{\partial y_{tt}} = 1 \quad \text{and} \quad \frac{\partial \phi}{\partial y_{xx}} = -a^2 \tag{25}$$

Using the definition in Eq. (22b), the partials can be interchanged as

$$\frac{\partial y_{tt}}{\partial q} = \frac{\partial^2}{\partial t^2} \left(\frac{\partial y}{\partial q}\right) = \nu_{tt} \quad \text{and} \quad \frac{\partial y_{xx}}{\partial q} = \frac{\partial^2}{\partial x^2} \left(\frac{\partial y}{\partial q}\right) = \nu_{tt} \tag{26}$$

Substituting Eqs. (25) and (26) into Eq. (24), we have

$$\nu_{tt} - a^2\nu_{xx} + f(x,t) = 0 \tag{27}$$

If we compare the definitions of variation in Eq. (2a) with the definition of sensitivity relationship in Eq. (22a), we have

$$\delta y(x,t) = \mu\eta(x,t) = (\delta q)\,\nu(x,t) \tag{28}$$

which gives

$$\eta(x,t) = \nu(x,t) \tag{29a}$$

and

$$\delta q = \mu \tag{29b}$$

Thus Eq. (27) becomes

$$\eta_{tt} - a^2\eta_{xx} + f(x,t) = 0 \tag{30}$$

which gives the PDE of the variations of the original system.

If we compare Eq. (30) with Eq. (23), we see that the variation $\eta(x,t) = \mu^{-1}\delta y(x,t)$ in Eq. (30) takes the place of the function y in Eq. (23) with $q = 1$. Therefore, the PDE for the variations is unchanged except by a scale factor. Thus the solution of the variation $\delta y(x,t)$ has the same form as that of the original function y.

Similarly for the adjoint system, one can obtain

$$\delta\bar{y}(x,t) = \bar{\mu}\bar{\eta}(x,t) = (\delta\bar{q})\,\bar{\nu}(x,t) \tag{31}$$

$$\bar{\eta}(x,t) = \bar{\nu}(x,t) \tag{32a}$$

$$\delta\bar{q} = \bar{\mu} \tag{32b}$$

$$\bar{\eta}_{tt} - a^2\bar{\eta}_{xx} + \bar{f}(x,t) = 0 \tag{33}$$

which is the PDE of the variations of the adjoint system.

THE WEAK EXTREMUM

Let us assume that the variations are separable in space x and time t, i.e.,

$$\eta(x,t) = f(x)g(t) \quad \eta(x,t_b+t_o-t) = f(x)g(t_b+t_o-t) \tag{34a}$$

$$\eta_t(x,t) = f(x)g_t(t) \quad \eta_t(x,t_b+t_o-t) = f(x)g_t(t_b+t_o-t) \tag{34b}$$

$$\eta_x(x,t) = f_x(x)g(t) \quad \eta_x(x,t_b+t_o-t) = f_x(x)g(t_b+t_o-t) \tag{34c}$$

Substituting Eq. (34) into Eq. (19b) gives

$$P(x,t) = f^2(x)g_t(t)g_t(t_b+t_o-t) + a^2f_x{}^2(x)g(t)g(t_b+t_o-t) \tag{35}$$

with the initial conditions from Eqs. (18a) and (18b)

$$\bar{\eta}(t_b) = \eta(t_o) = 0 \Rightarrow g(t_o) = 0 \tag{36a}$$

$$\bar{\eta}_t(t_b) = -\eta_t(t_o) = 0 \Rightarrow g_t(t_o) = 0 \tag{36b}$$

Let us construct the function g(t) with $t_o = 0$, and $\omega > 0$, as

$$g(t) = 1 - \cos \omega t \ , \ g(t_o) = 0 \tag{37a}$$

and

$$g_t(t) = \omega \sin \omega t \ , \ g_t(t_o) = 0 \tag{37b}$$

which satisfy Eq. (23). Then we have

$$g_t(t) \geq 0 \quad \text{if} \quad 0 \leq \omega t < \omega t_b < \pi \tag{38a}$$

$$g_t(t_b-t) \geq 0 \quad \text{if} \quad 0 \leq \omega(t_b-t) < \omega t_b < \pi \tag{38b}$$

$$g(t) \geq 0 \quad \text{if} \quad 0 \leq \omega t < \omega t_b \leq 2\pi \tag{39a}$$

and

$$g(t_b-t) \geq 0 \quad \text{if} \quad 0 \leq \omega(t_b-t) < \omega t_b < 2\pi \tag{39b}$$

Under these conditions the second variation is positive semi-definite, i.e.,

$$P(x,t) \geq 0 \quad , \quad \text{if } \omega(t_b-t_o) < \pi \tag{40a}$$

$$\text{or } t_b - t_o < \pi/\omega \tag{40b}$$

The above $g_t(t)$ is continuous and requires close neighborhood of the function y_t. This is a weak extremum. However, actual solutions of the wave equation show there are jumps in the functions y_t and y_x. This can only be

proved by a strong extremum later in the report. Let us assume in Eq. (34a) that

$$f(x) = (x_b - x)^2 (x - x_o)^2 \geqslant 0 \quad \text{for } x_o \leqslant x \leqslant x_b \tag{41}$$

which satisfies all possible boundary conditions for the variations of the wave equation. It is also noted that for the chosen g(t) in Eq. (37), the variation $\delta y(x,t)$ in Eq. (7a) and its partials obtained from Eqs. (2a) and (34a) are always arbitrarily positive for $t_b < \pi/\omega$. This also satisfies the fundamental lemma of the calculus of variations which gives the Euler-Lagrange Eq. (8a). A similar approach follows that $\delta y(x,t) \geqslant 0$ in Eq. (7c) for the derivation of Eq. (8b).

PERIOD OF THE WAVE EQUATION

It is not true that, "For a vibrating string there is no time interval short enough to guarantee that y(x,t) actually minimizes the action functional," (ref 6). This conjecture was based on the series solutions of the wave equation, as a system of infinitely many coupled oscillators, with infinitely many natural frequencies, (i.e., $\omega \to \infty$). However, if the problem is approached from a different angle, the solution of the wave equation with no forcing terms is given as the sum of two wave fronts (ref 10):

$$y(x,t) = \lambda(x-at) + \rho(x+at) \quad , \quad \text{for } Q = 0$$

$$y(x,0) = \lambda(x) + \rho(x) = \phi(x)$$

$$\left.\frac{\partial y}{\partial t}\right|_{x-0} = -a\lambda_x(x) + a\rho_x(x) = \vartheta(x)$$

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[6] I. M. Gelfand and S. V. Formin, Calculus of Variations, Prentice-Hall, 1963, pp. 22, 34, 42, and 161.

[10] C. R. Wylie, Advanced Engineering Mathematics, McGraw-Hill, 1951, pp. 211-212.

Then
$$y(x,t) = \frac{1}{2}\left[\phi(x-at) + \phi(x+at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} \theta(s)ds \tag{42}$$

Let us take an example of a prismatic bar (ref 11) with the left end fixed and the right free. If the free bar is initially stretched by a constant force and suddenly released, the stress wave is travelling from right to left at a period $T = 4\ell/a$, where $\ell = x_b - x_o$ is the length of the bar. From Figure 2 one can write in terms of unit step functions u for the partial derivatives y_x and y_t, for the wave fronts at some intermediate location ξ and time ζ.

$$y_x(x,t=\zeta) = [u(x-0) - u(x-\xi)] \tag{43a}$$

$$y_t(x,t=\xi) = -a\{u(t-\zeta) - u[t - (\frac{T}{2} - \zeta)]\} \tag{43b}$$

$$\text{where the period } T = 4\ell/a \tag{43c}$$

$$\xi/\ell = |(T/4) - \zeta|/(T/4) \tag{43d}$$

The corresponding values of ξ and ζ are

$$\begin{gathered} \text{at } \xi = \ell,\quad \ell/2,\quad 0,\quad \ell/2, \text{ and } \ell \\ \text{Then } \zeta = 0,\quad T/8,\quad T/4,\quad 3T/8, \text{ and } T/2 \text{ respectively.} \end{gathered} \tag{44}$$

It is noted that the "physical frequency" of the system is

$$\omega_p = 2\pi/T = (\pi a)/(2\ell)$$

This frequency is a constant never approaching infinite for finite parameters a and ℓ.

THE STRONG EXTREMUM

For the variations of Eq. (43), shown in Figures 2 and 3, we let

$$\begin{aligned} \delta y_x(x,\zeta) &= y_x(x,\zeta,\mu) - y_x(x,\zeta) \\ &= \mu[u(x-0) - u(x-\xi)] \end{aligned} \tag{45}$$

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[11] L. S. Jacobsen and R. S. Ayre, Engineering Vibrations, McGraw-Hill, 1958, p. 473 (Figures 10-18).

for $0 < \xi < \ell$,

$$\delta y_x(x,\zeta) = 0 \quad \text{at the right end}$$

$$\delta y(x,\zeta) = 0 \quad \text{at the left end}$$

Thus Eq. (45) satisfies the required boundary conditions. Similarly,

$$\delta y_t(x,\zeta) = -a\mu\{u(t-\zeta) - u[t-T/2)]\} \tag{46}$$

for $\zeta > 0$,

$$\delta y_t(x,\zeta) = 0 \quad \text{and} \quad \delta y(x,\zeta) = 0 \quad \text{initially}$$

It is noted that both δy_x and δy_t are constant within the intervals. Thus Eq. (46) satisfies the required initial conditions. Equations (45) and (46) fulfill the "general conditions" for the variation and are subject to discontinuity of the partial derivatives, where the step functions occur. This leads to a strong extremum.

We use the increment in time for computation by the finite element method as

$$t_b - t_o = \sigma = nT/4 \tag{47}$$

where $n = 1$ or 2.

Under this condition if t varies from t_o to $(t_o+\sigma)$, then $\sigma - t$ varies from $(t_o+\sigma)$ to t_o. We further divide the increment into four equal intervals, each interval having $T/16$ in time.

Since δy_x and δy_t are constants or zeroes, the second variation as given by Eq. (19) can be changed into finite summation instead of integration over the time domain as,

$$(\delta^2 J)_k = (\Delta t/2)\int_{x_o}^{x_b} P_k(x,t)dx + (\Delta t)\sum_{i=1}^{N-1}\int_{x_o}^{x_b} P_{k+i}(x,t)dx$$
$$+ (\Delta t/2)\int_{x_o}^{x_b} P_{k+N}(x,t)dt \tag{48}$$

where k is the index of advance and N is the number of intervals. For the interval of T/16 and with the aid of Eq. (47), the quantity N becomes N = 4 if n = 1 and N = 8 if n = 2.

From Eq. (19b) we have

$$P_{k+i}(x,t) = \delta y_t[x,(t_o+kT/16+iT/16)]\,\delta y_t[x,(t_o+kT/16+\sigma-iT/16)]$$
$$+ a^2\delta y_x[x,(t_o+kT/16+iT/16)]\,\delta y_x[x,(t_o+kT/16+\sigma-iT/16)] \tag{49}$$
$$\text{for } i = 0,1,2,3,\ldots N$$
$$k = 0,1,2,3,\ldots$$

If we integrate the above products in the spatial domain for n = 1, we obtain the the following as shown in Figures 4(a) and 4(b).

$$\int_{x_o}^{x_b} P_k(x,t)dx = 0 \quad \text{for all k} \tag{50a}$$

$$\int_{x_o}^{x_b} P_{k+4}(x,t)dx = 0 \quad \text{for all k} \tag{50b}$$

$$\int_{x_o}^{x_b} P_{k+1}(x,t)dx = a^2\mu^2\ \ell/2 > 0 \quad \text{for all k} \tag{50c}$$

$$\int_{x_o}^{x_b} P_{k+3}(x,t)dx = a^2\mu^2\ \ell/2 > 0 \quad \text{for all k} \tag{50d}$$

and

$$\int_{x_o}^{x_b} P_{k+2}(x,t)dx = a^2\mu^2\ell > 0 \quad \text{for all k} \tag{50e}$$

Substituting Eq. (49) into Eq. (48) gives

$$(\delta J^2)_{n=1} = (\Delta T)[0 + a^2\ell/2 + a^2\ell/2]\,\mu^2(\Delta t) > 0 \tag{51}$$

For n = 1, i.e., the increment σ being (T/4) the quarter period, the second variation is positive semi-definite. Thus, the variational method is valid and the functional yields a minimum.

The integration of Eq. (49) in the spatial domain for $n = 2$ (i.e., $\sigma = T/2$) gives the following for all k as shown in Figures 5(a) and 5(b).

$$\int_{x_o}^{x_b} P_k(x,t)dx = -a^2\mu^2\ell \tag{52a}$$

$$\int_{x_o}^{x_b} P_{k+1}(x,t)dx = -a^2\mu^2\ell/2 \tag{52b}$$

$$\int_{x_o}^{x_b} P_{k+2}(x,t)dx = 0 \tag{52c}$$

$$\int_{x_o}^{x_b} P_{k+3}(x,t)dx = a^2\ell\mu^2/2 \tag{52d}$$

and

$$\int_{x_o}^{x_b} P_{k+4}(x,t)dx = a^2\ell\mu^2/2 \quad \text{for all } k \tag{52e}$$

Substituting Eq. (52) into Eq. (47) gives

$$(\delta^2 J)_{n=2} = 0 \quad \text{for all } k \tag{53}$$

For $n = 2$, i.e., the increment σ being (T/2) the half period, the second variation is zero. Thus the variational method is on the margin to yield a minimum for the functional. It can be proved that for $\sigma > T/2$ the second variation becomes negative. Thus the variational method is valid only if the increment σ used for the finite element method is

$$\sigma \leqslant T/2 \tag{54}$$

where T is the period corresponding to the "physical frequency" of the system.

CONCLUSIONS

The functional in bilinear form is established and the Euler-Lagrange equations are derived to obtain the wave equation and its symmetrical adjoint equation. The second variation of the functional is given in terms of the

variations of the partials in space and time, both in the original variable and the adjoint variable. The adjoint system is defined as the image reflection in the time domain of the original system. The physical frequency of the system is determined (not series expansion frequency). There is an upper bound for the increment in time that can be used for computation purposes. The strong extremum was illustrated by an example of the wave equation where the first partials of the variables and their variations were subject to discontinuities.

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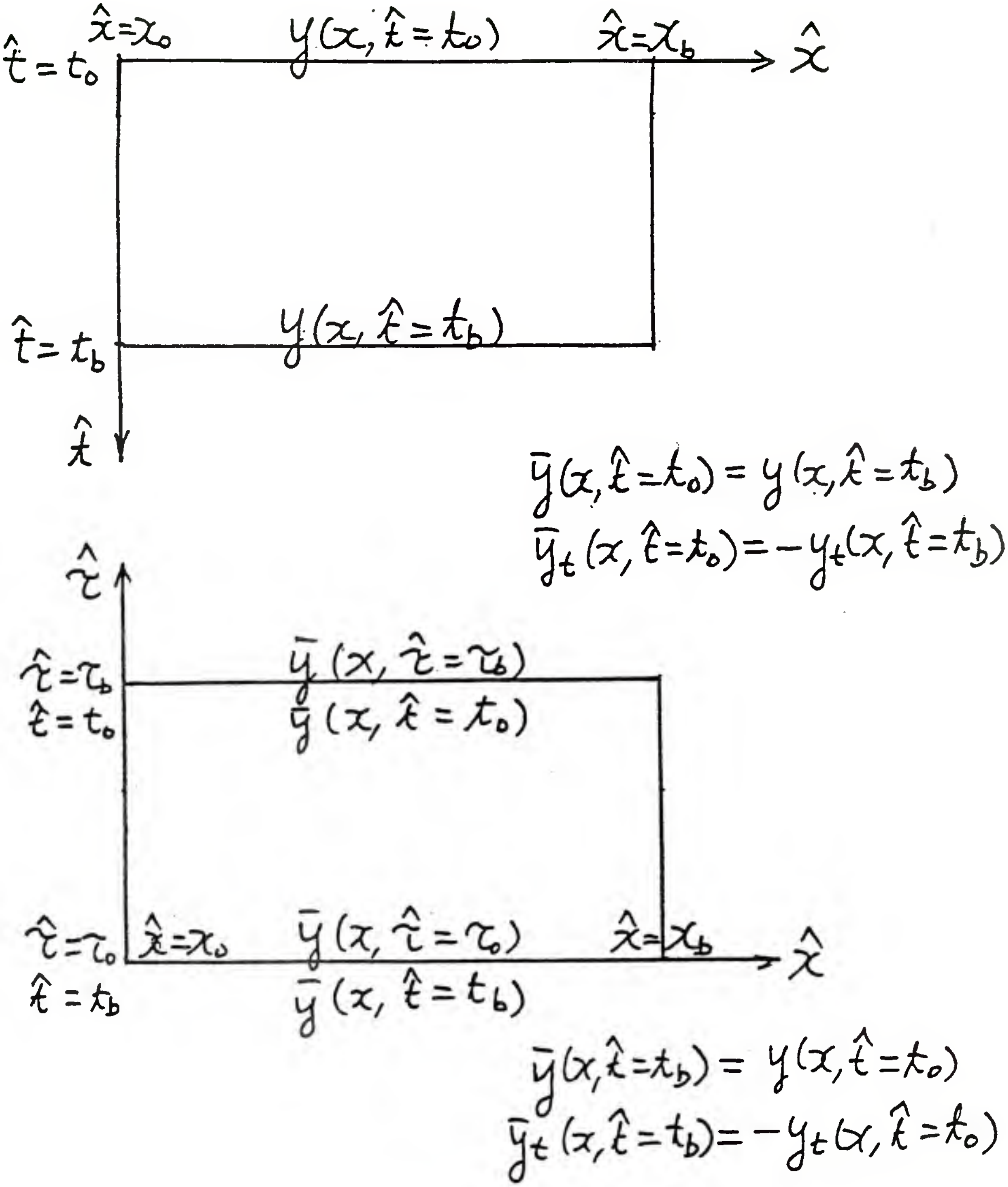


Figure 1. Image Reflection of the Adjoint System.

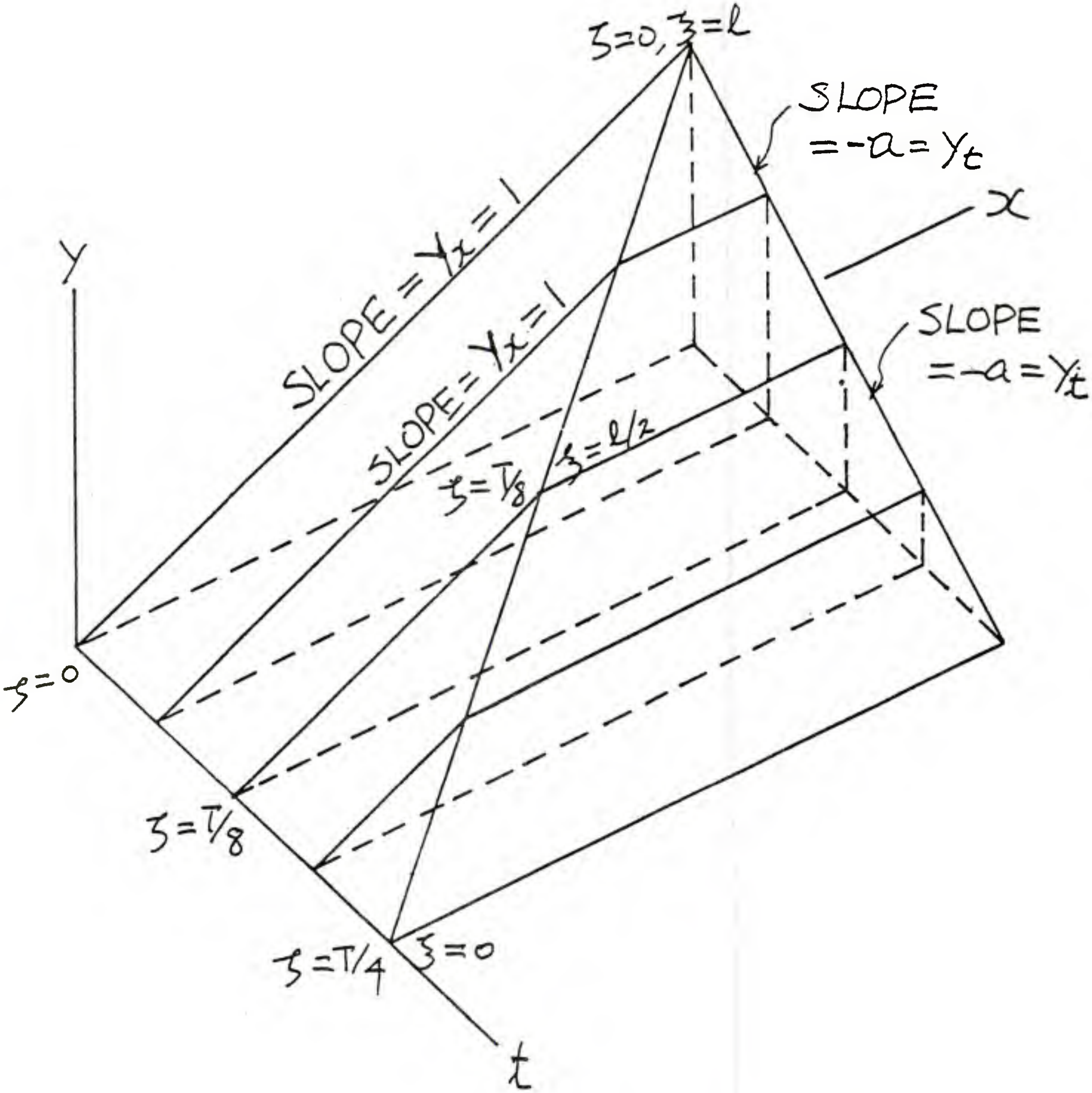


Figure 2. Spatial-Temporal Response of a Prismatic Bar With Sudden Release of Load at One End.

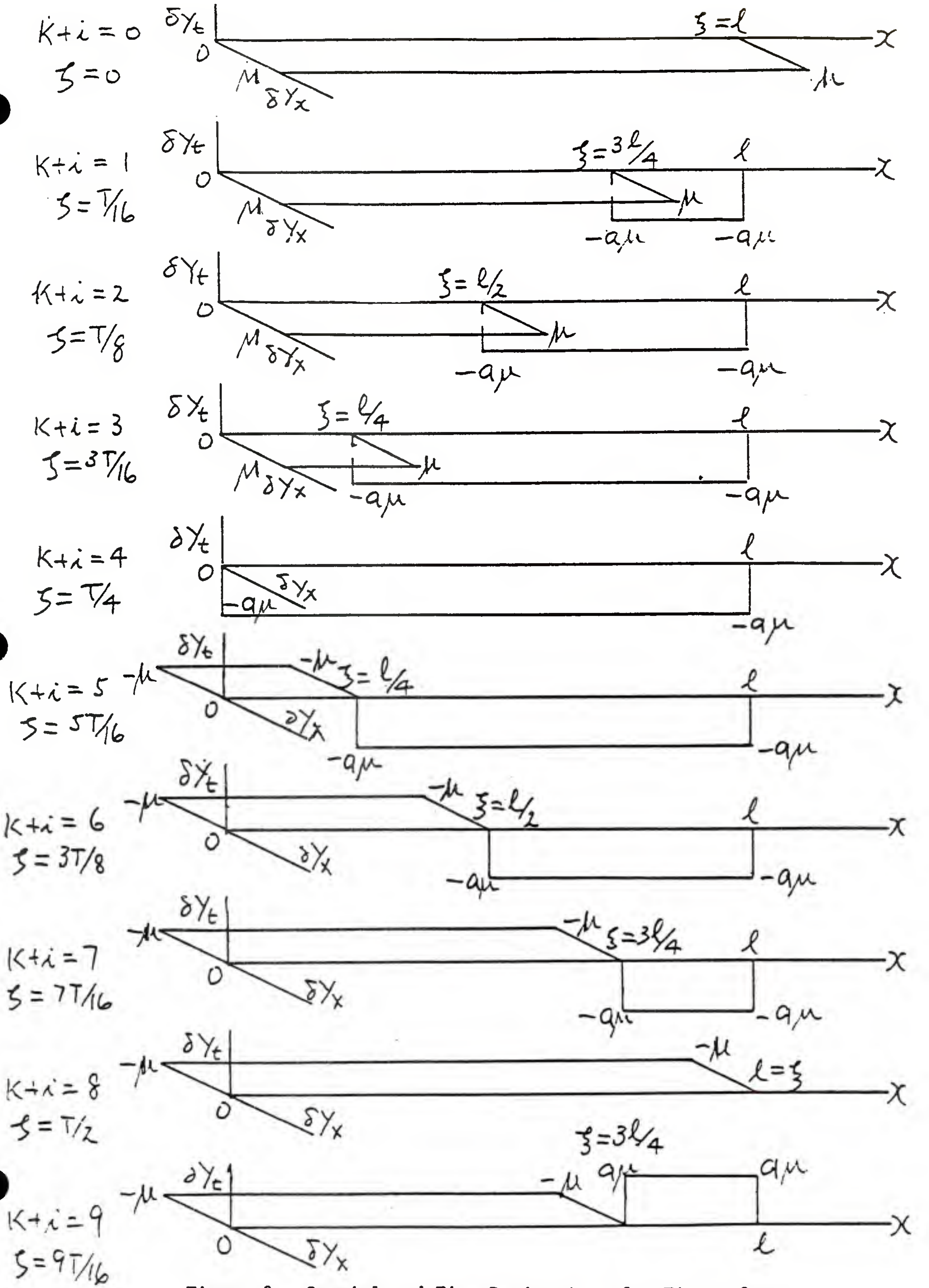


Figure 3. Spatial and Time Derivatives for Figure 2.

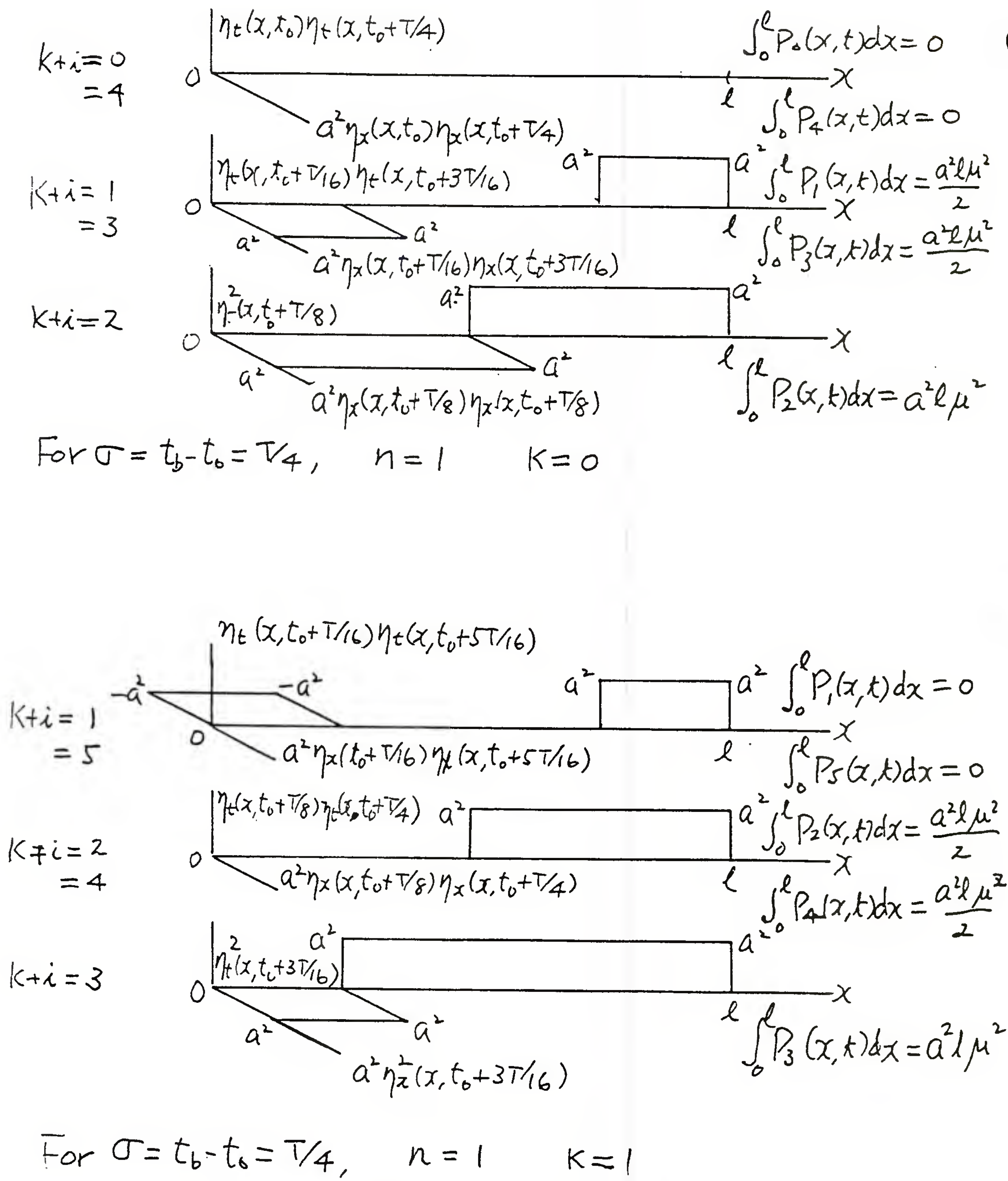
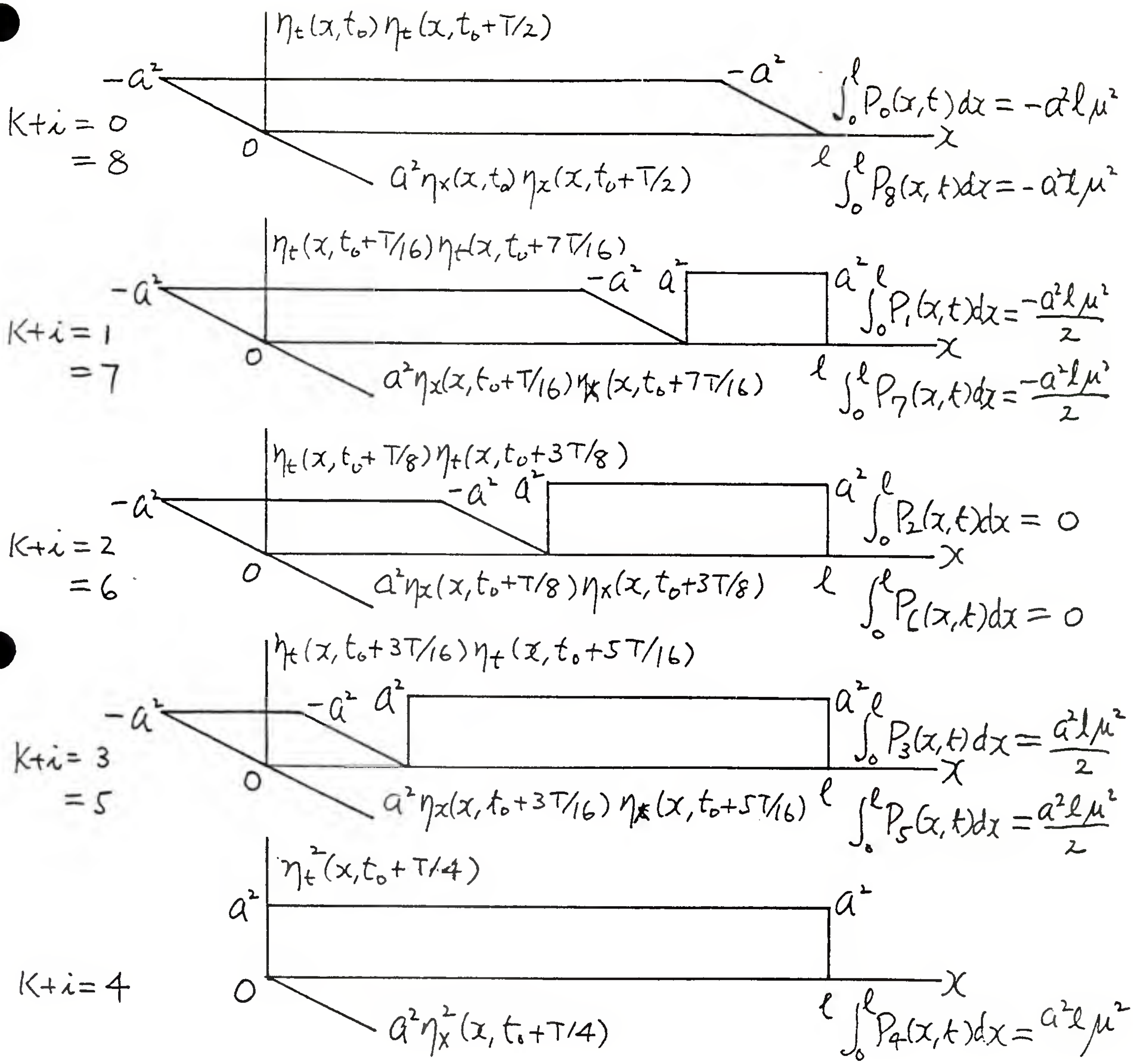


Figure 4. Integral of Products of Spatial and Time Derivatives Using Quarter Period Increment for Finite Element Method.



For $\sigma = t_b - t_o = T/2, \quad n=2, \quad K=0$

Figure 5(a). Integral of Products of Spatial and Time Derivatives Using Half Period Increment For Finite Element Method (with k=0).

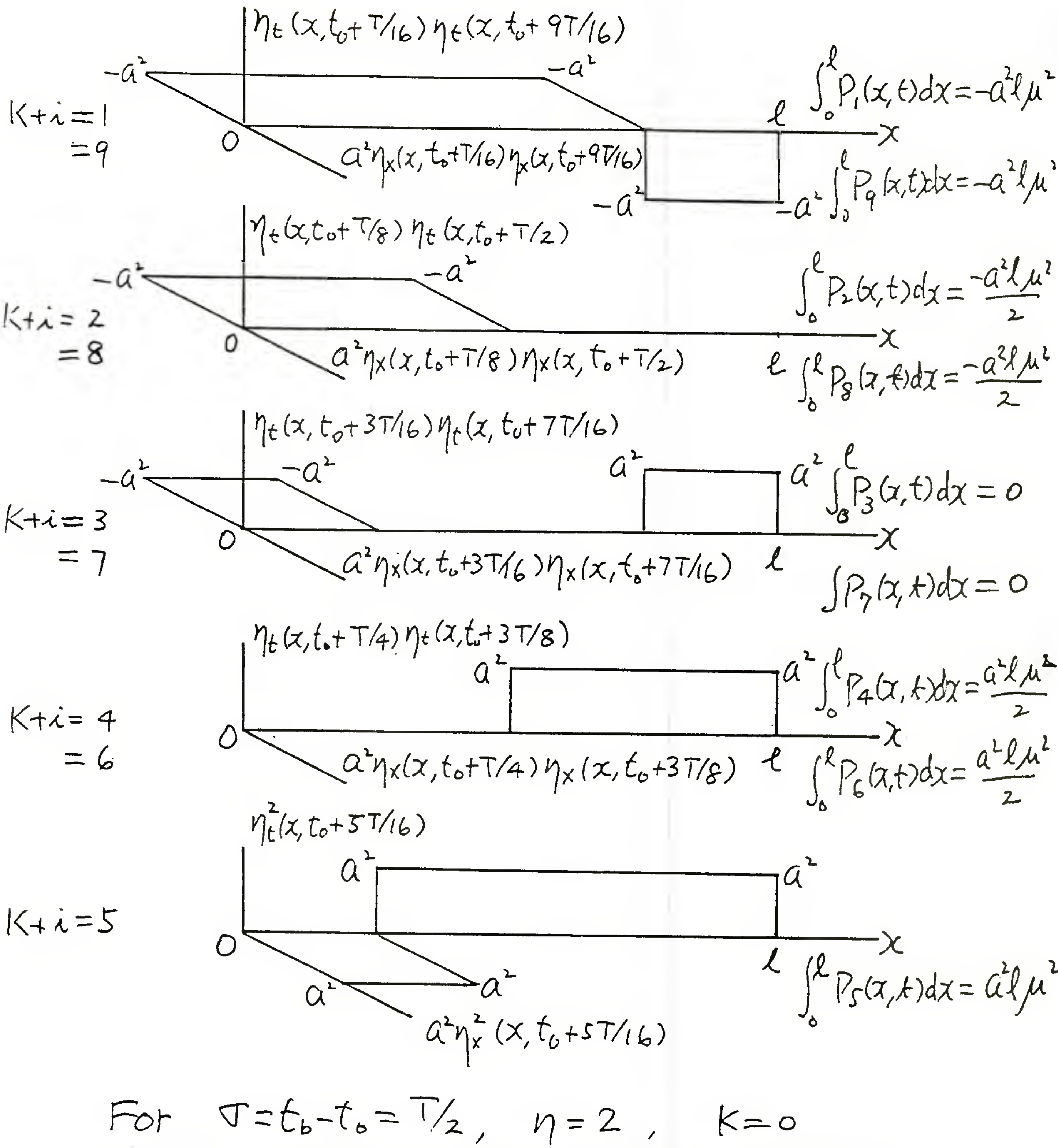


Figure 5(b). Integral of Products of Spatial and Time Derivatives Using Half Period Increment For Finite Element Method (with k=1).

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